### L E C T U R E 13

**Repetition (recurrence) of trials. The Bernoulli formula**

If several trials are made and the probability of an event *A* for each trial doesn’t depend on outcomes of other trials, such trials are called *independent from the event A*.

At various independent trials an event *A* can have either different probabilities or the same probability. We will further consider only such independent trials in which the event *A* has the same probability.

We use below the notion of *complex event* meaning by that overlapping of several events, which are called *simple*.

### Let *n* independent trials will be made in each of which an event *A* can either appear or not to appear. Assume that the probability of the event *A* for each trial is the same, namely equals *p*. Consequently, the probability of non-happening the event *A* in each trial is also constant and equals *q = 1 – p*.

Let’s pose the problem of calculating the probability that for *n* trials the event *A*

will happen exactly *k* times and consequently will not happen *n – k* times. It is

important to underline that it is not required the event *A* repeated exactly *k* times

in a certain sequence. For example, if the speech is about appearance of the

event *A* three times in four trials then the following complex events are possible:

*AAA Ā, AA Ā A, A Ā AA, Ā AAA*. The entry *AAA Ā* denotes that the event *A* happened at the first, second and third trials and it didn’t happen at the fourth trial, i.e. the opposite event *A* happened; the rest entries have the corresponding sense.

Denote the required probability by *Pn(k)*. For example, the symbol *P5(3)* denotes

the probability that the event will happen exactly 3 times for 5 trials and

consequently it will not happen 2 times.

### One can solve the posed problem by means of such-called Bernoulli formula.

### *Deduction of the Bernoulli formula:* The probability of one complex event consisting in that for *n* trials the event *A* will happen *k* times and will not happen *n – k* times is equal by the theorem of multiplication of probabilities of independent events to *pkqn – k*. There can be such complex events as much as combinations of *n* elements on *k* elements can be composed, i.e. . Since

### Pn(k) = p k q n – k

*Example.* The probability that the expense of electric power during one day will not exceed the established norm is equal to *p =* 0,75. Find probability that at the closest 6 days the expense of electric power will not exceed the norm for 4 days.

*Solution:* The probability of normal expense of electric power during each of 6 days is constant and equals 0,75. Consequently, the probability of overexpenditure of electric power for each day is also constant and equals *q = 1 – p = 1 – 0,75 = 0,25.* The required probability by the Bernoulli formula is equal to

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*The most probable number k*0 of occurrences of an event in independent trials is determined from the double inequality:

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where *n* – number of trials, *p* – probability of occurrence of the event in one trial, *q* – probability of non-occurrence of the event in one trial.

*Example.* 40 boxes of glass products have been delivered on a warehouse. The probability that all products of a randomly taken box appear intact is equal to 0,9. Find the most probable number of boxes in which all products appears intact.

*Solution*: By the hypothesis *n* = 40, *p* = 0,9, *q* = 0,1. The most probable number of boxes which are not containing damaged products is determined by the double inequality: ****or ****Therefore the required most probable number *k*0 = 36,9 .

**Local theorem of Laplace**

It easy to see that using the Bernoulli formula for great values *n* is sufficiently difficult because for example if *n = 50, k = 30, p = 0,1* then for finding the probability *P50(30)* it is necessary to calculate the expression

P50(30)=50!/(20!30!) \*(0,1) 30 (0,9) 20.

One arises the question: is it possible to calculate probability interested for us without using the Bernoulli formula? Yes, it can. Local theorem of Laplace gives the asymptotic formula which allows approximately to find the probability of appearance of an event exactly *k* times for *n* trials if the number of trials is sufficiently great.

**Local theorem of Laplace:** If the probability *p* of appearance of an event *A* for each trial is constant and differs from 0 and 1 then the probability *Pn(m)* that the event *A* will appear for *n* trials exactly *m* times is approximately equal to (the more precise, the greater *n*) the value of the function

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for  ******

There are tables with values of the function corresponding to positive values of the argument *x*. We use the same tables for negative values of the argument because the function is even, i.e  *f(-x)=f(x)*.

Thus, the probability that the event *A* will happen for *n* independent trials exactly *m* times is approximately equal to

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where  ******

*Example*. Find the probability that an event *A* will happen exactly 80 times for 400 trials if the probability of appearance of the event for each trial equals 0,2.

*Solution:* By the hypothesis, *n = 400; k = 80; p = 0,2; q = 0,8.*

Use the asymptotic formula of Laplace:

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Calculate the value *x* determined by data of the problem:

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By the table we find *f(0)=0*,3989. ******

The Bernoulli formula gives almost the same result: ******.

**Integral theorem of Laplace**

Suppose that *n* trials are made in each of which the probability of appearance of the event *A* is constant and equals *p* (*0 < p < 1*). How can we calculate probability *Pn(k1, k2)* that the event *A* will appear for *n* trials no less than *k*1 and no more than *k*2 times (for shortness, we will say “from *k*1 up to *k*2 times”)?

**Theorem.** If probability *p* of appearance of an event *A* for each trial is constant and differs from 0 and 1 then the probability *Pn(k1, k2)* that the event *A* will appear for *n* trials from *k*1 up to *k*2 times is approximately equal to the definite integral

***,*** (\*)

***where,  and , .***

*Example.* The probability that a detail has not passed a checking by the quality department, is equal to 0,2. Find the probability that among 400 randomly selected details appears unchecked from 70 up to 100 details.

*Solution:* By the hypothesis, *p = 0,2; q = 0,8; n = 400; k1 = 70; k2 = 100*.

Use the integral theorem of Laplace:

Calculate the lower and the upper limits of integration:

Thus, we have

By the table we find: .

The required probability 0.8882.

**Glossary**

**complex event** – сложное событие; **overlapping** – совмещение

**expense** – расход; **electric power** – электроэнергия

**overexpenditure** – перерасход

**quality department** – отдел технического контроля

**warehouse** – товарный склад; **intact** – целый (неповрежденный)